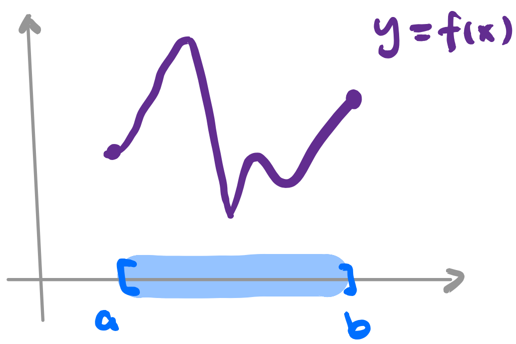
MATH 2050 - Continuous functions on intervals

(Reference: Bartle §5.3)

Q: What can we say about cts fcn $f : \underset{\text{A closed \& bdd interval}}{[a,b]} \to \mathbb{R}$?



Note: All points $c \in [a,b]$ are cluster points of $[a,b]$.

i.e. $\lim_{x \to c} f(x) = f(c)$

In terms of ε-δ def$^{=}$.

$\forall c \in [a,b]$, $\forall \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon, c) > 0$

s.t. $|f(x) - f(c)| < \varepsilon$ when $|x - c| < \delta$, $x \in [a,b]$

Recall: f cts at $c \Rightarrow f$ is "locally bdd" near c

Boundedness Thm: Any cts $f : [a,b] \to \mathbb{R}$ is bdd (globally on $[a,b]$)

i.e. $\exists\, M > 0$ st $|f(x)| \leq M \quad \forall x \in [a,b]$.

Proof: Argue by contradiction. Suppose f is NOT bdd on $[a,b]$.

$\Rightarrow \forall n \in \mathbb{N}$, $\exists\, x_n \in [a,b]$ st $|f(x_n)| > n$ ...... (*)

We obtain a seq. (x_n) in $[a,b]$, hence is bdd.

By Bolzano-Weierstrass Thm, $\exists$ convergent subseq. (x_{n_k}) of (x_n)

say $\lim_{k \to \infty} (x_{n_k}) =: x_*$

Now, $a \leq x_{n_k} \leq b \quad \forall k \in \mathbb{N} \overset{\text{limit}}{\underset{\text{thm}}{\Longrightarrow}} a \leq \lim (x_{n_k}) \leq b$

ie $x_* \in [a,b]$.

$\lim_{x \to x_*} f(x) = f(x_*)$

By continuity of f at x_*, and seq. criteria, → "$\lim_{x \to x_*} f(x) = \lim_{k \to \infty} (f(x_{n_k}))$"

$$\lim_{k \to \infty} f(x_{n_k}) = f\left(\lim_{k \to \infty} x_{n_k}\right) = f(x_*)$$

So, $f(x_{n_k}) \to f(x_*)$ as $k \to \infty$ $\Rightarrow$ $(f(x_{n_k}))$ is bdd.

; contradiction

However, $\underbrace{|f(x_{n_k})| > n_k}_{\text{by construction } (*)} \geq k \quad \forall k \in \mathbb{N}$ $\Rightarrow$ $(f(x_{n_k}))$ is unbdd.

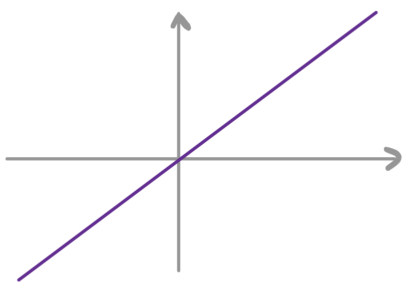
$\blacksquare$

Remark: All assumptions are required in the theorem.

(1) unbdd interval

$f: \mathbb{R} \to \mathbb{R}$

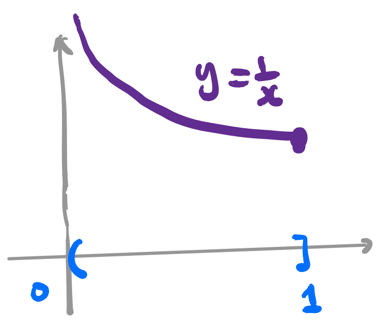
$f(x) := x$



(2) non-closed interval

$f: (0, 1] \to \mathbb{R}$

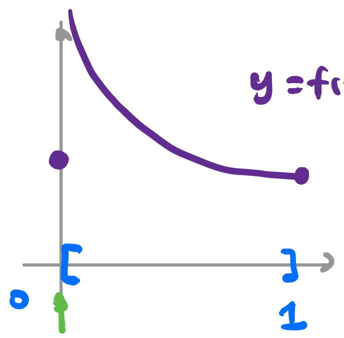
$f(x) := \frac{1}{x}$



(3) not continuity

$f: [0, 1] \to \mathbb{R}$

$$f(x) := \begin{cases} 1/x, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0. \end{cases}$$



f is NOT cts at 0

By Boundedness Theorem, $\exists$ exist in $\mathbb{R}$

$$M := \sup\{f(x) \mid x \in [a, b]\}$$

$$m := \inf\{f(x) \mid x \in [a, b]\}$$

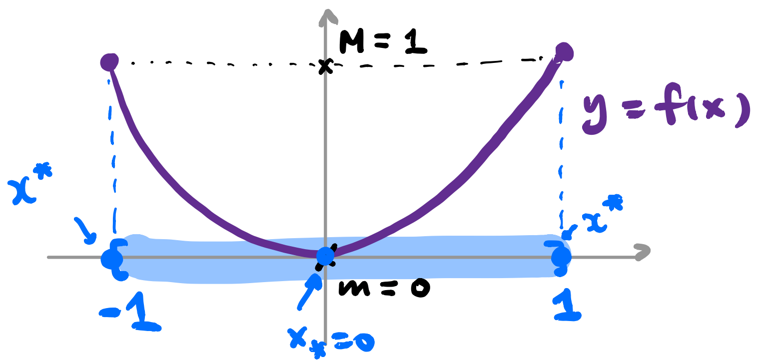
Extreme Value Thm: A cts $f: [a,b] \to \mathbb{R}$ always achieve its maximum and minimum, i.e.

$\exists\ x^* \in [a,b]$ st $f(x^*) = M := \sup\{f(x) \mid x \in [a,b]\}$

not nec. unique

$\exists\ x_* \in [a,b]$ st $f(x_*) = m := \inf\{f(x) \mid x \in [a,b]\}$

Example: $f(x) = x^2$, $f: [-1,1] \to \mathbb{R}$



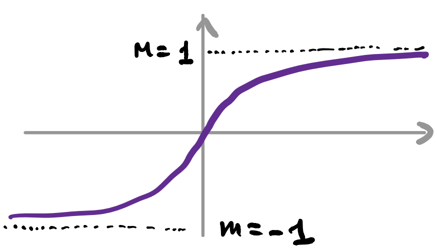
Caution: There can be more than one maxima x^* and minima x_*.

Remarks: All assumptions are required.

(1) unbdd interval

$f: \mathbb{R} \to \mathbb{R}$

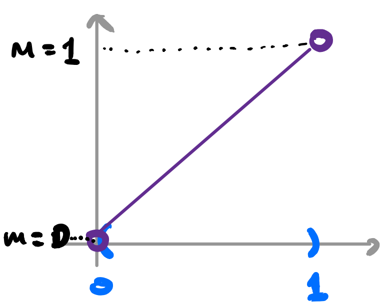
$f(x) = \tanh x$



(2) non-closed interval

$f: (0,1) \to \mathbb{R}$

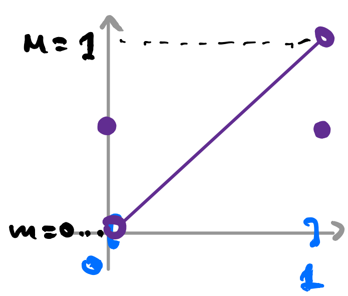
$f(x) = x$



(3) not cts

$f: [0,1] \to \mathbb{R}$

$f(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 1/2 & \text{if } x = 0, 1 \end{cases}$



Proof: We only prove the existence of x^*.

Since $M := \sup\{f(x) \mid x \in [a,b]\}$, $\forall \varepsilon > 0$, $\exists\, x_\varepsilon \in [a,b]$ st.

$$M - \varepsilon < f(x_\varepsilon)$$

Take $\varepsilon = \frac{1}{n}$, then we obtain a sequence $(x_n) \subseteq [a,b]$ st.

$$M - \frac{1}{n} < f(x_n) \le M$$

By Bolzano-Weierstrass Thm, since (x_n) is a bdd seq.

$\Rightarrow \exists$ convergent subseq. (x_{n_k}) of (x_n), say $\underset{\substack{\uparrow \\ [a,b]}}{x^*} := \lim (x_{n_k})$

Claim: $f(x^*) = M$

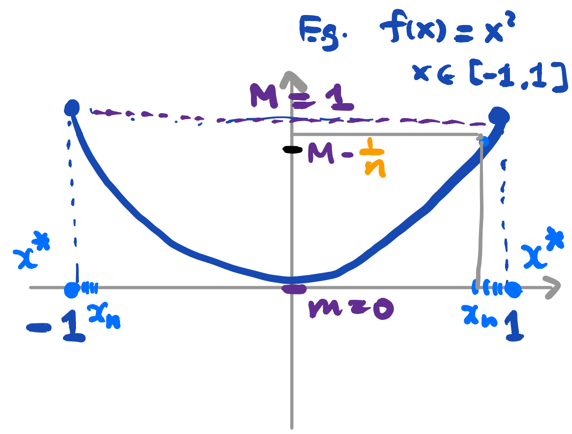
Pf: Since $M - \frac{1}{n_k} < f(x_{n_k}) \le M$

for all $k \in \mathbb{N}$,

take $k \to \infty$, by continuity of f at x^*

$$M \le f(x^*) = \lim_{k\to\infty} f(x_{n_k}) \le M$$

Limit theorems



$\square$

Three important theorems about continuous $f : [a,b] \to \mathbb{R}$

- Boundedness Thm
- Extreme Value Theorem.
- Intermediate Value Theorem

[compactness]

<u>Extreme Value Thm</u>: A cts $f : [a,b] \to \mathbb{R}$ always achieve its absolute maximum and minimum, i.e.

$\exists\ x^* \in [a,b]$ s.t. $f(x^*) = M := \sup\{f(x) \mid x \in [a,b]\}$

$\exists\ x_* \in [a,b]$ s.t. $f(x_*) = m := \inf\{f(x) \mid x \in [a,b]\}$

not nec. unique

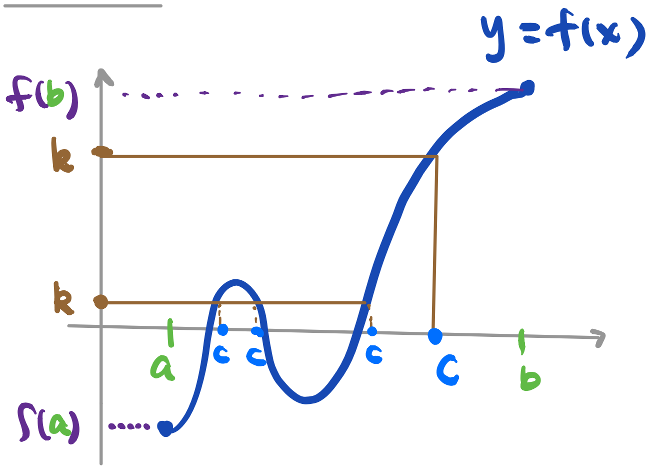
<u>Intermediate Value Theorem</u> [connectedness]

Let $f : [a,b] \to \mathbb{R}$ be a cts function s.t. $f(a) < f(b)$.

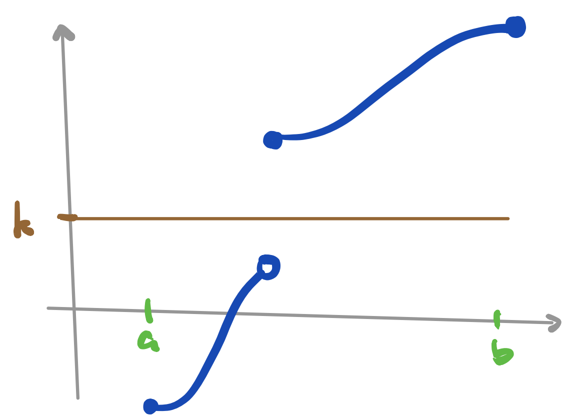
THEN, $\forall\ k \in (f(a), f(b))$, $\exists\ c \in [a,b]$ s.t.

$$f(c) = k$$

<u>Picture:</u>

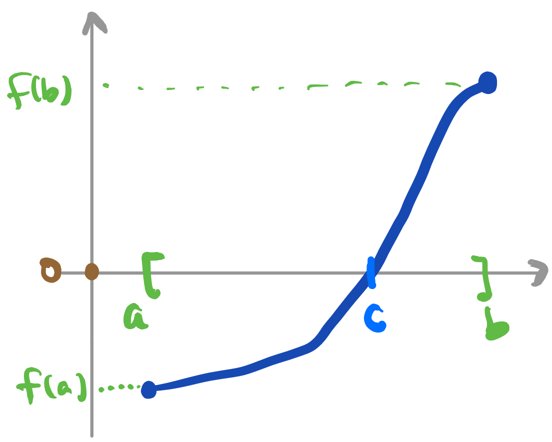


<u>Continuity is needed</u>



Proof: It suffices to consider the case:

$$f(a) < 0 < f(b) \quad \text{and} \quad k = 0$$



[$\because$ The general case follows by considering $g(x) := f(x) - k$.]

$g(c) = 0 \iff f(c) = k$

Q: How to locate a "root" where $f(c) = 0$?

A: Method of bisection.

Define a nested seq of closed & bdd intervals I_n as follows.

Take $I_1 := [a, b] =: [a_1, b_1]$

Consider the midpt. $\frac{a_1+b_1}{2}$ of I_1

Case 1: $f\left(\frac{a_1+b_1}{2}\right) < 0 \Rightarrow$ take $I_2 := [a_2, b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$

Case 2: $f\left(\frac{a_1+b_1}{2}\right) > 0 \Rightarrow$ take $I_2 := [a_2, b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$

Case 3: $f\left(\frac{a_1+b_1}{2}\right) = 0 \Rightarrow$ DONE, take $c = \frac{a_1+b_1}{2}$.

Repeat this process for I_2.

Either you locate a root (Case 3), or you obtain a seq. of closed & bdd intervals $I_n := [a_n, b_n]$. $\text{Length}(I_{n+1}) = \frac{1}{2}\text{Length}(I_n)$

st $\begin{cases} I_{n+1} \subseteq I_n \quad \forall n \in \mathbb{N} & \text{nested} \\ f(a_n) < 0 < f(b_n) \quad \forall n \in \mathbb{N}. & \text{———— } (\#) \end{cases}$

By Nested Interval Property, $\bigcap_{n=1}^{\infty} I_n = \{c\}$

($\because$ $\text{Length}(I_n) \to 0$)

Claim: $f(c) = 0$.

Pf: Since $\lim(a_n) = \lim(b_n) = c$, take $n \to \infty$ in (*),
by continuity of f at c,

$$f(c) \le 0 \le f(c), \text{ ie } f(c) = 0$$

$\square$